No. XV.

Solution of a General Case of the Simple Pendulum. By Eugenius Nulty.—Read 21st August, 1818.

IN a letter which I wrote to Dr. Patterson, and which the Society thought worthy of publication, I found a new converging series for determining the times of oscillation of the simple pendulum in a plane. This has since led me to consider a more general case of the simple pendulum, in which the motion is supposed to arise from the action of gravity and an impulse not directed in a vertical plane, and to take place in the surface of the sphere of which the radius is the length of the string connecting the oscillating point and centre of suspension. As the series which I have found in solving this problem have not been noticed by the latest writers on mechanics, I have thought that the following investigation might not be unworthy the attention of the Society.

Let x, y, z be the vertical and horizontal distances of the oscillating point from three rectangular planes X. Y, Z, given in position with respect to the centre of suspension, dt the element of time during which the motion is considered as uniform, and g the accelerating force of gravity.

The velocities at the beginning of dt in the directions of x, y, z are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$; the forces lost during this element are

therefore $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$; and by the principle of virtual velocities, the sum of these forces, multiplied by the variations of their directions, is equal to the action of gravity multiplied by the variation of its direction. We have therefore

$$\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z=g\delta x. \qquad (1)$$

Let the invariable length of the pendulum be denoted by a, and let the common intersections of the three planes **X**, **Y**, and **Z** be in the vertical passing through the centre of suspension at a distance equal to a below this point. We shall then have

$$\sqrt{\{(a-x)^2+y^2+z^2\}}=a,$$

and of which the variation is

$$\frac{x-a}{a}\delta x+\frac{y}{a}\delta y+\frac{z}{a}\delta z=0, \qquad (2)$$

an expression which the variations in the equation (1) must satisfy.

Let us therefore multiply this expression by an arbitrary quantity T, and add the result to the equation (1). We shall then find

$$\left(\frac{d^2x}{dt^2}+T\frac{x-a}{a}-g\right)\delta x+\left(\frac{d^2y}{dt^2}+T\frac{y}{a}\right)\delta y+\left(\frac{d^2z}{dt^2}+T\frac{z}{a}\right)\delta z=0.$$

The variations δx, δy, δz are independent in this equation by virtue of the arbitrary quantity T, and accordingly we have

$$\frac{d^2x}{dt^2}+T\frac{x-a}{a}-g=0,$$

$$\frac{d^2y}{dt^2}+T\frac{y}{a}=0,$$

$$\frac{d^2z}{dt^2}+T\frac{z}{a}=0,$$

which are easily verified by observing that the first terms are the components of the inertia of the oscillating point in the directions of x, y, z; the second terms, the components of the tension of the string in the same directions; and g the effect of gravity in the vertical direction x.

Multiplying these equations respectively by dx, dy, dz, adding the results, and observing that by virtue of the equation (2) we have

$$\frac{x-a}{a}dx+\frac{y}{a}dy+\frac{z}{a}dz=0,$$

we shall get

$$\frac{d^2x}{dt^2}dx+\frac{d^2y}{dt^2}dy+\frac{d^2z}{dt^2}dz=gdx,$$

of which the integral is

$$\frac{dx^2+dy^2+dz^2}{dt^2}=2g(b-x). \tag{3}$$

Again, multiplying the second and third of the preceding equations by z and y, and taking the difference of the results, we shall find

$$z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}=0,$$

of which the integral is

$$\frac{z\mathrm{d}y - y\mathrm{d}z}{\mathrm{d}t} = b'. \tag{4}$$

The equation (3) expresses the known principle of living forces, and the equation (4) that of the equable description of areas.

Let v be the horizontal angle formed by the projection of a on the plane X, and by the vertical plane Z. Then it is evident that $y = \sqrt{(2ax - x^2)}.\sin.\, v$ and $z = \sqrt{(2ax - x^2)}.\cos.\, v$; and their differentials are

$$\mathrm{d}y = \frac{a - x}{\sqrt{(2ax - x^2)}}.\sin.v\mathrm{d}x + \sqrt{(2ax - x^2)}\cos.v\mathrm{d}v,$$

$$\mathrm{d}z = \frac{a - x}{\sqrt{(2ax - x^2)}}.\cos.v\mathrm{d}x - \sqrt{(2ax - x^2)}\sin.v\mathrm{d}v.$$

Substituting these expressions in the equations (3) and (4), we shall find after obvious reduction,

$$\frac{a^2\mathrm{d}x^2 + (2ax - x^2)\mathrm{d}v^2}{\mathrm{d}t^2} = 2g(2ax - x^2).(b - x),$$

$$\frac{(2ax - x^2)\mathrm{d}v}{\mathrm{d}t} = b',$$

from the first of which eliminating dv, and from the second dt, we get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm \frac{(2g)^{\frac{1}{2}}}{a}\sqrt{F}, \tag{5}$$

$$\frac{\mathrm{d}x}{\mathrm{d}v} = \pm \frac{1}{a.c^{\frac{1}{2}}}.(2ax - x^2)\sqrt{F}, \tag{6}$$

in which $F = x^3 - (2a + b)x^2 + 2abx - c$, and $c = \frac{b'}{2g}$.

These are the equations from which the motion of the pen-

dulum is to be determined. The first will give the time in a vertical direction, and the second the horizontal angle described about the vertical which passes through the centre of of suspension.

In order to integrate these equations, let us observe that the oscillating point can neither ascend so high on the spherical surface as to attain a point of which the vertical ordinate is equal to $2a$, nor descend so low as the horizontal, tangential plane X. The curve described will therefore be contained between two horizontal circles drawn on the spherical surface, and it will evidently touch these circles in two points P and Q, corresponding to the greatest and least values of the vertical ordinate x. Let these values, or the ordinates of P and Q be denoted by p and q, and let us observe that the vertical velocity $\frac{dx}{dt}$ decreases as the oscillating point approaches P and Q, and vanishes at the instant of their coincidence. We shall therefore have $F = 0$, both when $x = p$ and $x = q$, and accordingly this equation must be divisible by each of the factors $x - p$, $x - q$, and consequently by their product $x^2 - (p+q)x + pq$. Let the third factor of F be $x - r$. Then we shall have

$$F \text{ or } x^3 - (2a+b)x^2 + 2abx - c = \{x^2 - (p+q)x + pq\}.(x - r),$$

from which, by comparing like powers of x, we get

$$2a + b = p + q + r, \quad 2ab = pq + (p+q)r, \quad c = pqr,$$

and thence

$$r = 2a + \frac{pq}{2a - p - q}, \quad c = (2a + \frac{pq}{2a - p - q})pq.$$

The equation F may now be transformed into the product

$$\{x^2 - (p+q)x + pq\} \cdot \{x - (2a + \frac{pq}{2a - p - q})\},$$

which may be written in the form,

$$\left\{\left(x-\frac{p+q}{2}\right)^2-\left(\frac{p-q}{2}\right)^2\right\}.\left\{\left(x-\frac{p-q}{2}\right)-\left(2a+\frac{pq}{2a-p-q}-\frac{p-q}{2}\right)\right\}$$

$$=\left(\frac{p-q}{2}\right)^2.(1-u^2).\frac{p-q}{2k}.(1+ku), \text{ by putting}$$

$$\frac{p+q}{2}-x=\frac{p-q}{2}u, \text{ and } 2a+\frac{pq}{2a-p-q}-\frac{p-q}{2}=\frac{p-q}{2k}.$$

By virtue of this assumption, we have

$$\surd F=\frac{p-q}{2}.\left(\frac{p-q}{2k}\right)^{\frac{1}{2}}.\surd\left\{(1-u^2).(1+ku)\right\},$$

in which $u=-1$ corresponds to $x=p$, and $u=1$ to $x=q$.

Substituting this expression in the equation (5), and observing that $\mathrm{d}x=-\frac{p-q}{2}\mathrm{d}u$, we shall find

$$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{[g(p-q)]^{\frac{1}{2}}}{ak^{\frac{1}{2}}}\surd\left\{(1-u^2).(1+ku)\right\}, \text{ and thence}$$

$$\mathrm{d}t=\frac{ak^{\frac{1}{2}}}{[g(p-q)]^{\frac{1}{2}}}.\frac{\mathrm{d}u}{\surd\left\{(1-u^2).(1+ku)\right\}}. \qquad (7)$$

With respect to the equation (6), we have

$$2ax-x^2=(2a-x)x=\left(2a-\frac{p+q}{2}+\frac{p-q}{2}u\right).\left(\frac{p+q}{2}-\frac{p-q}{2}u\right)$$

$$=\frac{(p-q)^2}{4k'k''}.(1+k'u).(1-k''u), \text{ by assuming}$$

$$\frac{p-q}{2}=k'\left(2a-\frac{p+q}{2}\right)=k''\frac{p+q}{2}, \text{ and accordingly}$$

$$\frac{du}{dv}=\frac{1}{a.c^{\frac{1}{2}}}\cdot\frac{(p-q)^2}{4k'k''}.(1+k'u).(1-k''u).\surd\{(1-u^2).(1+ku)\},$$

and thence

$$dv=\left(\frac{2ck}{p-q}\right)^{\frac{1}{2}}.\frac{4ak'k''}{(p-q)^2}.\frac{du}{(1+k'u).(1-k''u).\surd\{(1-u^2).(1+ku)\}}. \quad (8)$$

These are the simplest forms in which dt and dv can be presented. It is impossible, however, to express their integrals in finite terms by means of circular arcs and logarithms; but series exhibiting their approximate values may be found in the following manner:

Putting, for the sake of brevity, the constant coefficient $\frac{ak}{g(p-q)}=K$, and expanding the factor $(1+ku)^{-\frac{1}{2}}$ by the binomial theorem, we have

$$t=K^{\frac{1}{2}}\int\frac{du}{\surd(1-u^2)}\left\{1-\frac{1}{2}ku+\frac{1.3}{2.4}k^2u^2-\frac{1.3.5}{2.4.6}k^3u^3+\frac{1.3.5.7}{2.4.6.8}k^4u^4-\&c.\right\},$$

in which, if we put the integral of $\frac{du}{\surd(1-u^2)}=\text{arc}(\sin.=x)=A$, and the integral of $\frac{udu}{\surd(1-u^2)}=-\surd(1-x)=A'$, we shall have

$$\int\frac{u^2du}{\surd 1-u^2}=\frac{A-u\surd 1-x^2}{2}=\frac{B}{2},$$

$$\int\frac{u^4du}{\surd 1-u^2}=\frac{3B-2u^3\surd 1-u^2}{2.4}=\frac{C}{2.4},$$

$$\int\frac{u^6du}{\surd 1-u^2}=\frac{5C-2.4u^5\surd 1-u^2}{2.4.6}=\frac{D}{2.4.6},$$

&c. &c.

$$\int \frac{u^3 du}{\sqrt{1-u^2}} = \frac{A'-u^2\sqrt{1-u^2}}{3} = \frac{B'}{3},$$

$$\int \frac{u^5 du}{\sqrt{1-u^2}} = \frac{B'-3u^4\sqrt{1-u^2}}{3.5} = \frac{C'}{3.5},$$

$$\int \frac{u^7 du}{\sqrt{1-u^2}} = \frac{C'-3.5u^6\sqrt{1-u^2}}{3.5.7} = \frac{D'}{3.5.7},$$

&c. &c.

by virtue of which, the integral of the preceding expression is

$$t=K^{\frac{1}{2}}\left\{A+\frac{3B}{4^2}k^2+\frac{5.7C}{4^2.8^2}k^4+\frac{7.9.11D}{4^2.8^2.12^2}k^6+\frac{9.11.13.15E}{4^2.8^2.12^2.16^2}k^8+\text{\&c.}\right.$$

$$\left.+\tfrac{1}{2}A'k+\frac{5B'}{2.4.6}k^3+\frac{7.9C'}{2.4.6.8.10}k^5+\frac{9.11.13D'}{2.4.6.8.10.12.14}k^7+\text{\&c.}\right\},$$

the difference between two values of which corresponding to given values h, h' of x will give the time taken by the oscillating point to describe a portion of the curve, corresponding to the vertical height $h-h'$.

If we extend the integral from $x=p$ to $x=q$, and consequently from $u=-1$ to $u=1$, we shall have the time (t) which the oscillating point requires to descend from the highest point P on the spherical surface to the lowest point Q, and vice versa. In this case, the integral A becomes simply $\frac{\pi}{2}$. (π being put for the semicircumference of a circle to radius $=1$,) the integrals B, C, D, &c. become respectively A, $3.A$, $D=3.5A$, &c. and the integrals A', B', C', D', &c. all vanish. We shall therefore have

$$(t)=K^{\frac{1}{2}}\pi\left\{1+\frac{1.3}{4^2}k^2+\frac{1.3.5.7}{4^2.8^2}k^4+\frac{1.3.5.7.9.11}{4^2.8^2.12^2}k^6+, \ \&c.\right\},$$

$$=K^{\frac{1}{2}}\pi\left\{1+\frac{1.3}{4^2}k^2+A_2\frac{5.7}{8^2}k^2+A_3\frac{9.11}{12^2}k^2+, \ \&c.\right\}, \qquad (9)$$

A_2, A_3, representing the second, third. &c. terms.

Let us resume the differential equation (7) and put $s=\frac{1-u^2}{1+ku}$. From this we get

$$u=-\frac{ks}{2}\pm\frac{1}{2}\surd(4-4s+k^2s^2)=-\frac{ks}{2}\pm\frac{1}{2}S^{\frac{1}{2}},$$

putting for the moment $S=4-4s+ks^2$, and thence $\surd\{(1-u^2).(1+ku)\}=s^{\frac{1}{2}}(1+ku)=\surd\left\{s\left(1-\frac{k^2s}{2}\pm\frac{k}{2}S^{\frac{1}{2}}\right)\right\}$, and $du=-\frac{ds}{S^{\frac{1}{2}}}\left\{1-\frac{ks}{2}\pm\frac{k}{2}S^{\frac{1}{2}}\right\}$. Substituting these expressions, we shall have

$$dt=K^{\frac{1}{2}}.\frac{-ds}{\surd\{s(4-4s+k^2s^2)\}}.$$

Let $2-ms$ and $2-ns$ be the factors of the quadratic $4-4s+k^2s^2$ in this expression. We shall then have $m+n=2$, $mn=k^2$, and thence $m=1+\surd(1-k^2)$, $n=1-\surd(1-k^2)$. The preceding expression may now be written

$$dt=K^{\frac{1}{2}}.\frac{-ds}{\surd\{s(2-ms).(2-ns)\}},$$

which will take the form of the original equation (7) by assuming $s=\frac{1-u'}{m}$; since then $-ds=\frac{du'}{m}$, $\surd\{s(2-ms).(2-ns)\}=$

$$\sqrt{\left\{\frac{(1-u'^2)}{m}.\frac{2m-n}{m}(1+\frac{n}{2m-n}u')\right\}}=\frac{K_1^{\frac{1}{2}}}{m}.\sqrt{\{(1-u'^2).(1+k_1u')\}},$$

by putting $\frac{n}{2m-n}=\frac{1-\sqrt{(1-k)}}{1+3\sqrt{(1-k^2)}}=k_1$, and $2m-n=1+3(\sqrt{1-k^2})=K_1$. We have therefore

$$dt=(\frac{K}{K_1})^{\frac{1}{2}}.\frac{du'}{\sqrt{\{(1-u'^2).(1+k_1u')\}}},$$

an expression which integrated between the limits $u'=1$ and $u'=-1$ as before, gives

$$(t)=2\pi.(\frac{K}{K_1})^{\frac{1}{2}}.\left\{1+\frac{1.3}{4^2}k_1^2+A_2\frac{5.7}{8^2}k_1^2+A_3\frac{9.11}{12^2}k_1^2+\&c.\right\}$$

a series exactly similar to that above found, and derivable from it by introducing the factor K_1, and changing k to k_1.

It is evident that a third series may be derived from this by introducing a new factor $K_2=\sqrt{\{1+3\sqrt{(1-k_1)}\}}$, and changing k_1 into $k_2=\frac{1+\sqrt{(1-k_1^2)}}{1+3\sqrt{(1\ \ k^2)}}$; and that a similar process may be continued ad infinitum. We shall therefore have

$$(t)=2^n\pi\ (\frac{K}{K_1.K_2.K_3.\ldots K_n})^{\frac{1}{2}}.\left\{1+\frac{1.3}{4^2}k_n^2+A_2.\frac{5.7}{8^2}k_n^2+\&c.\right\},$$

or rejecting the second, third, &c. terms of the series in the brackets as indefinitely small,

$$(t)=2^n\pi(\frac{K}{K_1.K_2.K_3.\ldots K_n}). \qquad (10)$$

This is an elegant expression for the determination of (t).

It has not been hitherto noticed, at least to my knowledge, and is the object I had in view in the present paper.

It may be proper to observe that the equation (10) may be easily modified so as to apply to the case in which the vibrating point moves in a vertical plane. We have merely to put $q=0$ in the expressions represented by K and k, since then the constant arbitrary quantity b' expressing the effect of the impulse from which the conical motion arises will vanish, and the vibrating point will descend to the horizontal plane X. The values of K and k will become, in this case, $\frac{a}{\sqrt{\{g(4a-p)\}}}$ and $\frac{p}{4a-p}$, and the corresponding value of (t) will be exactly of the same form as that just considered. If we limit the expression to the factor K, we shall have a particular case of the first expression found for (t), the same as that already given for the common pendulum.

With respect to the equation (10), I shall only observe that its integral may be found in a series by means of the integrals A, B, C, &c. A', B', C', &c. after the factors $(1+ku)$, $(1+k'u$, $(1-k''u)$, are expanded by the binomial theorem. By using peculiar artifices, other series may also be found; but their coefficients are so complicated, as to deter me from inserting their investigation.

I shall conclude this paper by a computation of the value of (t) for a particular value of k. This will give an idea of the rapidity with which the quantities k_1, k_2, &c. decrease, and at the same time show the great superiority of the formula (10) over the series (9).

The extreme value of which k is susceptible being unity, let us suppose this quantity greater than its mean value $\frac{1}{2}$, or $=\frac{3}{5}$. Then we have $k_1=\frac{1}{17}$, $k_2=\frac{1}{2263+}$, a fraction of which the square and higher powers may be safely neglect-

ed. We shall therefore have $K_1 = \frac{17}{5}$, $K_2 = \frac{67.911688}{17}$, $\sqrt{(K_1.K_2)} = 3.685422$, and accordingly

$$(t) = K^{\frac{1}{2}} \frac{4}{3.685422} = K^{\frac{1}{2}}(1.08535737),$$

a result which is true to the last figure.

Again, substituting $\frac{3}{5}$ instead of k in the series (9), we shall get

$$
\begin{aligned}
(t) = K^{\frac{1}{2}} \{ & 1.00000000 \} \\
& 6750000 \\
& 1328906 \\
& 328904 \\
& 90192 \\
& 26218 \\
& 7914 \\
& 2453 \\
& 775 \\
& 248 \\
& 81 \\
& 26 \\
& 8 \\
\hline
= K^{\frac{1}{2}} \{ & 1.08535725 \},
\end{aligned}
$$

in which the last two figures are incorrect, although the divisions have been extended to the ninth decimal place.